Limits of Polynomial Packings for $\mathbb{Z}_{p^k}$ and $\mathbb{F}_{p^k}$

Jung Hee Cheon
(Seoul National University & Crypto Lab Inc.)

Keewoo Lee
(Seoul National University)

Sketch

- **Formal & Unified Study of "Polynomial Packing"**
  - ... which appears in various contexts:
  - HE & SHE-based MPC (HE Packing), IT-MPC (RMFE), Correlation Extractor, ZK...

- **Upper Bounds & Impossibility Results**
  - Packing Density, Level-Consistency, & Surjectivity

- **Implications**
  - SHE-based MPC over $\mathbb{Z}_{2^k}$, HE Packing, RMFE

Definition

Definition

Polynomial Packing

$$R^n \quad (R = \mathbb{Z}_{p^k}, \mathbb{F}_{p^k}) \;\overset{\text{Pack}}{\underset{\text{Unpack}}{\rightleftarrows}}\; \mathcal{R} = \mathbb{Z}_{p^t}[x]/f(x)$$

* Packing Density = $\log(|R|^n) \,/\, \log(|\mathcal{R}|)$

Degree-D Packing

$$\begin{array}{ccc} R^n & \xrightarrow{\text{Pack}} & \mathcal{R} \\ \downarrow P(\cdot) & \circlearrowleft & \downarrow P(\cdot) \\ R^n & \xleftarrow{\text{Unpack}} & \mathcal{R} \end{array}$$

$P(\cdot)$: (Multivariate) Polynomial of Degree $\leq D$

Definition

Remark: Unpack may differ for each multiplicative level.

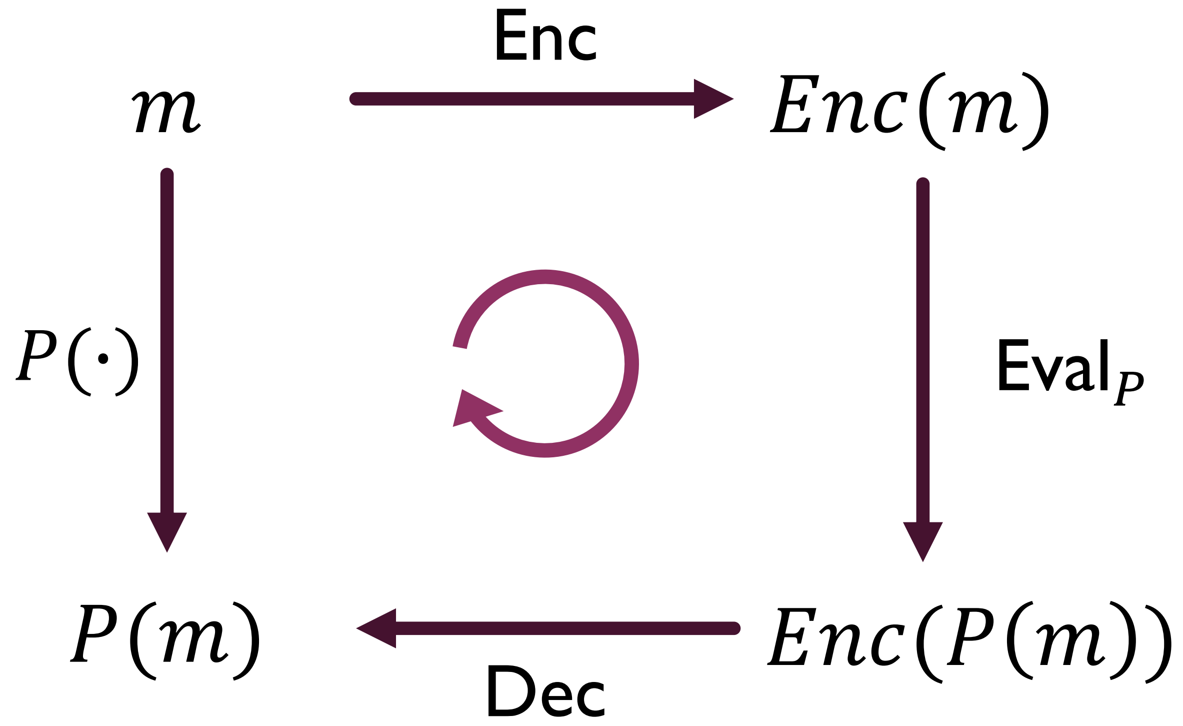
Definition 3.2 (Degree-D Packing). *Let $\mathcal{P} = (\mathsf{Pack}_i, \mathsf{Unpack}_i)_{i=1}^{D}$ be a collection of packing methods for R^n into $\mathcal{R}$. We call $\mathcal{P}$ a degree-D packing method, if it satisfies the following for all $1 \le i \le D$:*

- *If $a(x), b(x)$ satisfy $\mathsf{Unpack}_i(a(x)) = \boldsymbol{a}$, $\mathsf{Unpack}_i(b(x)) = \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in R^n$, then $\mathsf{Unpack}_i(a(x) \pm b(x)) = \boldsymbol{a} \pm \boldsymbol{b}$ holds;*
- *If $a(x), b(x)$ satisfy $\mathsf{Unpack}_s(a(x)) = \boldsymbol{a}$, $\mathsf{Unpack}_t(b(x)) = \boldsymbol{b}$ for $\boldsymbol{a}, \boldsymbol{b} \in R^n$ and $s, t \in \mathbb{Z}^+$ such that $s + t = i$, then $\mathsf{Unpack}_i(a(x) \cdot b(x)) = \boldsymbol{a} \odot \boldsymbol{b}$ holds.*

Contexts & Examples

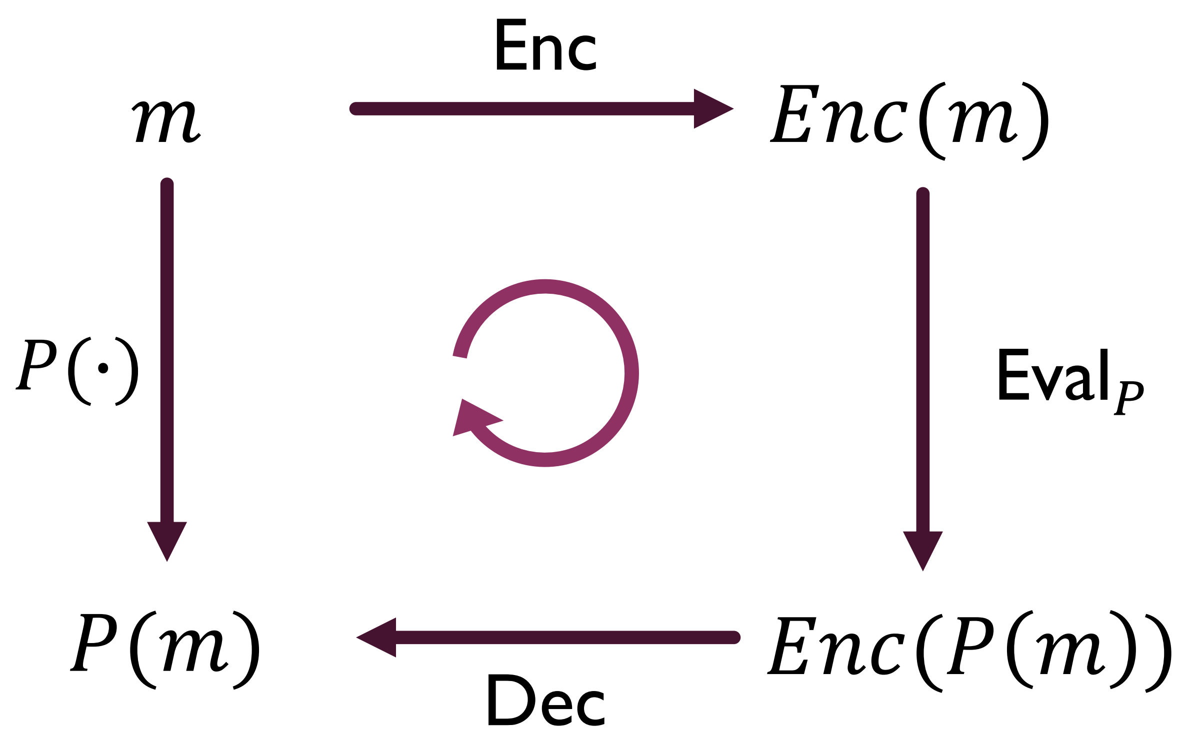
Homomorphic Encryption

- HE supports computation on encrypted data.



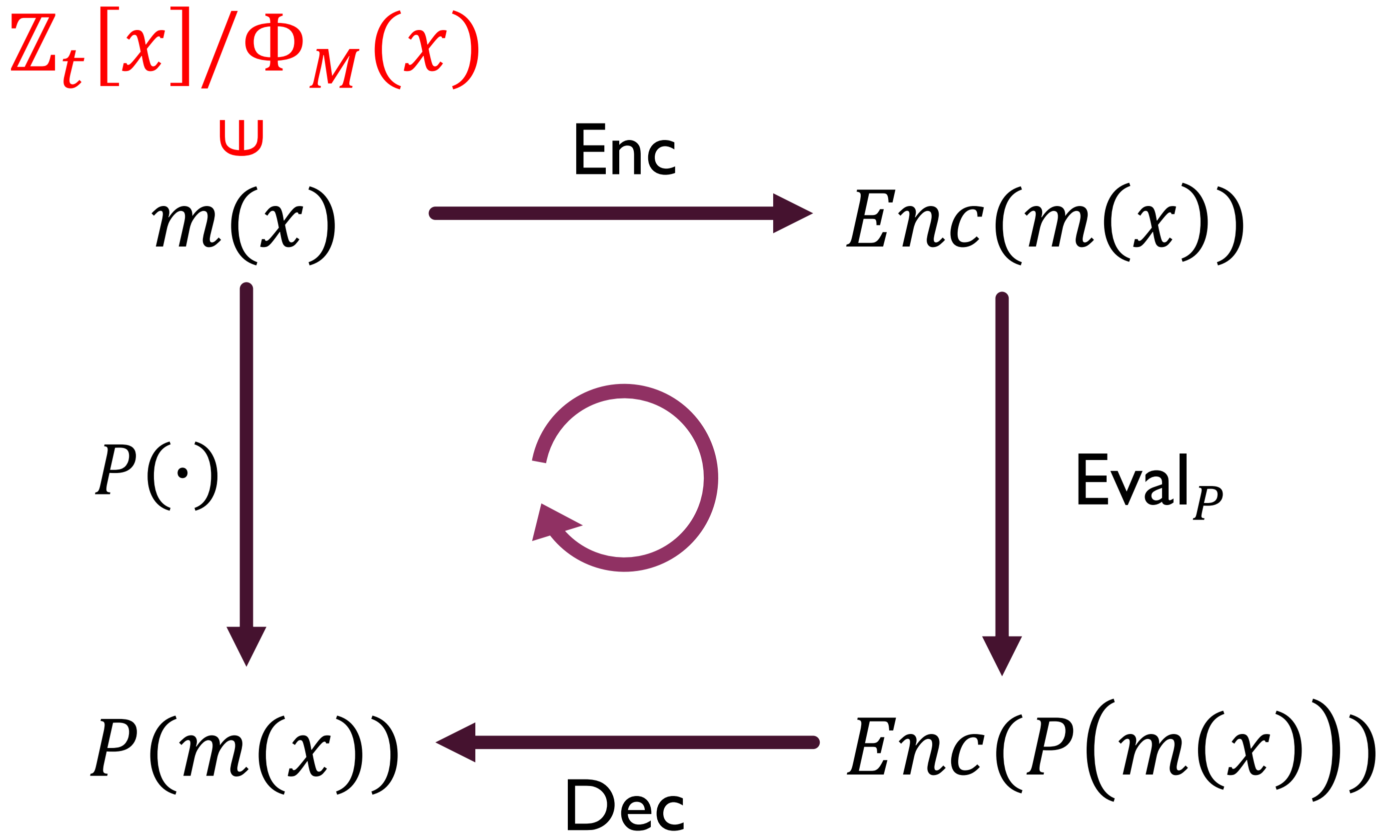
Homomorphic Encryption

- HE supports computation on encrypted data.
- Concurrent HE schemes are often based on RLWE for efficiency.
  - e.g. BGV, FV

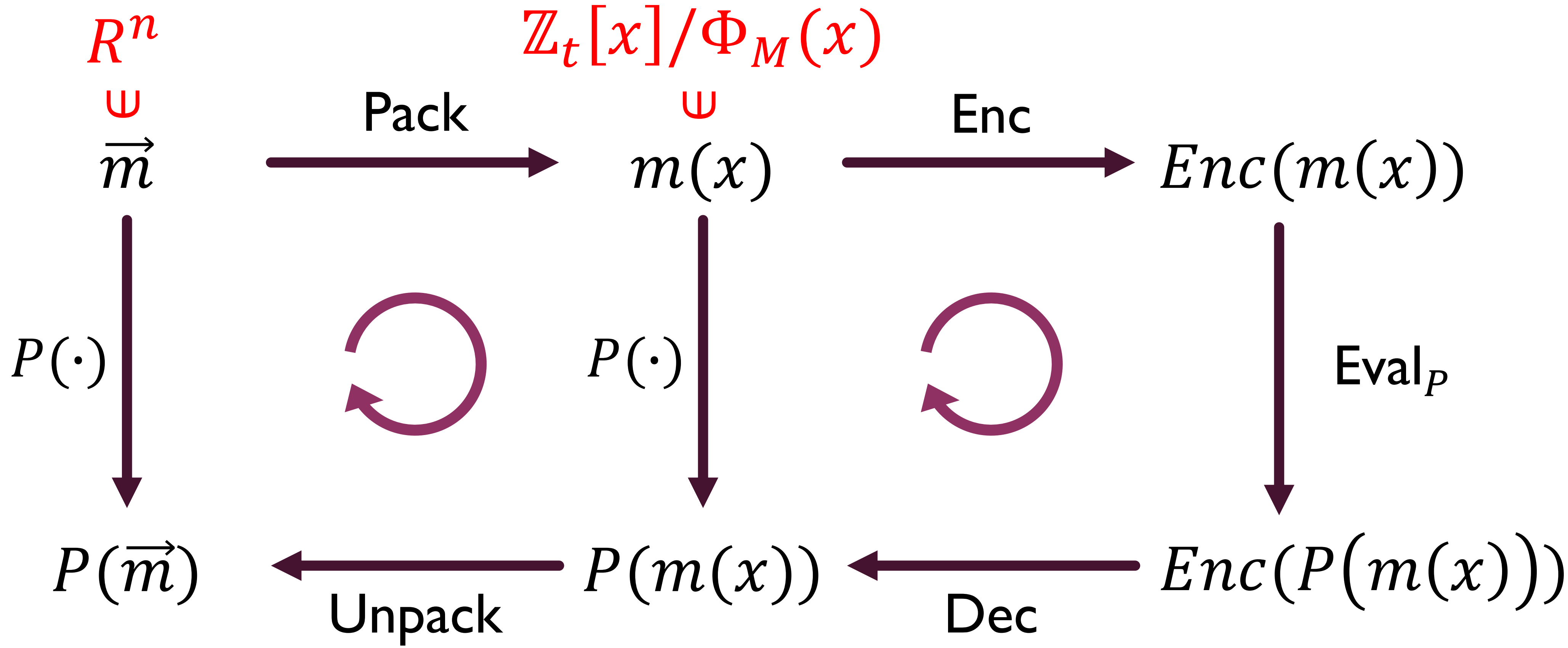


Homomorphic Encryption

- HE supports computation on encrypted data.
- Concurrent HE schemes are often based on RLWE for efficiency.
  - ➢ e.g. BGV, FV
  - ➢ Practical Usability?



HE Packing [Smart-Vercauteren;PKC10]



HE Packing: Examples

- $\Phi_M(x) = \prod_{i=1}^{r} F_i(x) \bmod p$ & $\deg F_i = d$

- Traditional Packing Method [Smart-Vercauteren;PKC10]
  - $(\mathbb{F}_{p^d})^r \xrightarrow{\cong} \mathbb{Z}_p[x]/\Phi_M(x)$: **Degree-∞**, Density $= 1$
  - $(\mathbb{Z}_p)^{\varphi(M)} \xrightarrow{\cong} \mathbb{Z}_p[x]/\Phi_M(x)$, if $\Phi_M(x)$ fully splits mod p : **Degree-∞**, Density $= 1$
- HELib Packing for $\mathbb{Z}_{p^k}$-messages [Gentry-Halevi-Smart;PKC12], [Halevi-Shoup;Eurocrypt15]
  - $(\mathbb{Z}_{p^k})^r \to \mathbb{Z}_{p^k}[x]/\Phi_M(x)$: **Degree-∞**, Density $= 1/d$
- Recent Developments in SHE-based MPC over $\mathbb{Z}_{2^k}$ (SPDZ-family)
  - Overdrive2k [Orsini-Smart-Vercauteren;CT-RSA20] : **Degree-2**, Density $\approx 1/5$
  - MHz2k [**Cheon**-Kim-**Lee**;Crypto21] : **Degree-2**, **Density** $\approx \mathbf{1/2}$

RMFE [Cascudo-Cramer-Xing-Yuan;Crypto18]

Using "Large Field" is often required due to:

1. **Mathematical Structures**
   - **Shamir Secret Sharing** : We can interpolate at most q points over $\mathbb{F}_q$
2. **Security**
   - **Linear MAC** : $MAC_\alpha(x) \coloneqq \alpha \cdot x$ over $\mathbb{F}_q$ has soundness error $1/q$

RMFE [Cascudo-Cramer-Xing-Yuan;Crypto18]

- **Reverse Multiplication-Friendly Embedding (RMFE)**
  - Embed algebraic structure of copies of small field (e.g. $\mathbb{F}_2^n$) into a larger field (e.g. $\mathbb{F}_{2^d}$).
  - Essentially, RMFEs are **Degree-2** packings from $\mathbb{F}_q^n$ into $\mathbb{F}_{q^d} \cong \mathbb{F}_q[x]/f(x)$.
  - Now a Standard Tool in IT-MPC (e.g. [DLN;Crypto19], [DLSV;Euro20], [PS;Euro21], …)
  - Also used in ZK (e.g. [BMRS;Crypto21], [CG;FC22])

Theorems & Implications

Packing Density

- **Theorem**
  - Roughly speaking, density of degree-D packing method $\lesssim 1/D$
  - For $d =$ [deg. of irreducible quotient poly.],

$$[\text{packing density}] \leq \frac{1}{D} + \frac{1}{d}\left(1 - \frac{1}{D}\right)$$

- **Implications**
  1. MHz2k [CKL;Crypto21] achieves near-optimal density (as a degree-2 packing for $\mathbb{Z}_{2^k}$)
  2. ($\mathbb{F}_{p^k}$ Version) New and more general proof for upper bound on rate of RMFE
  3. First upper bound on rate of RMFE over Galois rings [Cramer-Rambaud-Xing;Crypto21]

Level-Consistency

- **Motivation**
  - FHE, Homomorphic computation between different mult. levels (e.g. Reshare Protocol)
- **Theorem**
  - If level-consistency holds,

$n \le$ [# of distinct **irred.** factors of quotient poly. mod p]

- **Implications**

1. Optimality of HELib packing with respect to packing density and level-consistency
2. Impossibility of Efficient Level-Consistent HE Packing for $\mathbb{Z}_{2^k}$
3. Importance of "Constant Packing Trick" of MHz2k for Level-dependent packings

Surjectivity

- **Motivation**
  - Malicious "Packer" might leverage invalid packings in protocols.
- **Theorem**
  - If surjectivity holds,

$$n \leq [\ \text{\# of distinct } \textbf{linear} \text{ factors of quotient poly. mod } p^k\ ]$$

- **Implication**
  1. Impossibility of Surjective HE Packing for $\mathbb{Z}_{2^k}$
  2. Necessity of ZKPoMK in HE-based MPC over $\mathbb{Z}_{2^k}$ (First conceptualized in MHz2k)

Summary

- Formal & Unified Study of Polynomial Packing
  - which appears in various contexts:
  - HE & SHE-based MPC (HE Packing), IT-MPC (RMFE), Correlation Extractor, ZK…
- Upper Bounds & Impossibility Results
  - Packing Density, Level-consistency, and Surjectivity

Summary

- Implications on SHE-based MPC over $\mathbb{Z}_{2^k}$ (c.f. MHz2k [CKL;Crypto21])
  1. MHz2k achieves near-optimal packing density
  2. Importance of "Constant Packing Trick" of MHz2k for Level-dependent packings
  3. Necessity of ZKPoMK in HE-based MPC over $\mathbb{Z}_{2^k}$ (First conceptualized in MHz2k)
- Implication on HE Packing
  1. Optimality of HELib packing with respect to packing density and level-consistency
- Implications on RMFE
  1. New and more general proof for upper bound on rate of RMFE
  2. First upper bound on rate of RMFE over Galois rings (c.f. [CRX;Crypto21])

Conclusion

1. Packing is not a question asked before secure computation.
   - Messages are “static” (e.g. PKE): No need to worry about structure of messages.
2. Packing is a question shared by secure computation primitives.
   - Messages are “dynamic” (HE, MPC, ZK): Algebraic structure of messages matters.
3. There might be more questions of like this!
   - Especially when we try to apply secure computation to real-life problems.

Thank You!

* ePrint: ia.cr/2021/1033
* E-mail: activecondor@snu.ac.kr
* Webpage: keewoolee.github.io